Hyperuniformity on the Sphere

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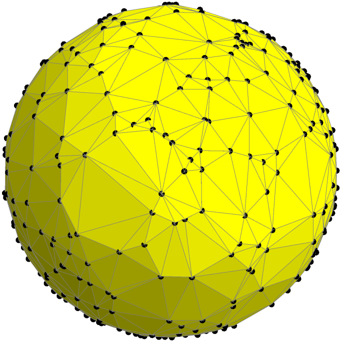
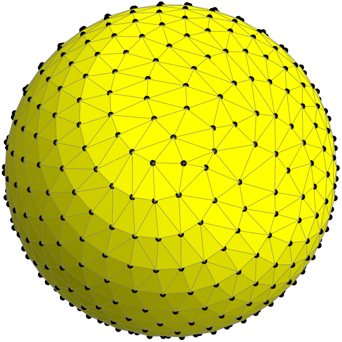
Two point distributions





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Then we can try to minimise or maximise these measures for given N.

Combinatorial measures

- discrepancy

$$D_N(X_N) = \sup_C \left| \frac{1}{N} \sum_{n=1}^{N} \chi_C(\mathbf{x}_n) - \sigma(C) \right|$$

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$$\delta_N(X_N) = \sup_{\mathbf{x} \in S^d} \min_k |\mathbf{x} - \mathbf{x}_k|$$

- separation

$$\Delta_N(X_N) = \min_{i \neq j} |\mathbf{x}_i - \mathbf{x}_j|$$

Analytic measures

- error in numerical integration

$$I_N(f, X_N) = \left| \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{x}_n) - \int_{S^d} f(\mathbf{x})\, d\sigma_d(\mathbf{x}) \right|$$

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- Worst-case error for integration in a normed space H:

$$\mathrm{wce}(X_N, H) = \sup_{\substack{f \in H \\ \|f\|=1}} I_N(f, X_N)),$$

L^2-discrepancy and energy

- L^2-discrepancy:

$$\int_0^\pi \int_{S^d} \left| \frac{1}{N} \sum_{n=1}^{N} \chi_{C(\mathbf{x},t)}(\mathbf{x}_n) - \sigma_d(C(\mathbf{x},t)) \right|^2 \, d\sigma_d(\mathbf{x})\, dt$$

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- (generalised) energy:

$$E_g(X_N) = \sum_{\substack{i,j=1\\ i\neq j}}^{N} g(\langle \mathbf{x}_i, \mathbf{x}_j \rangle) = \sum_{\substack{i,j=1\\ i\neq j}}^{N} \tilde{g}(\|\mathbf{x}_i - \mathbf{x}_j\|),$$

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L^2-discrepancy and the worst case error (for many function spaces) turn out to be generalised energies of the underlying point configuration.

Hyperuniformity in $\mathbb{R}^d$

Heuristic

Hyperuniformity = asymptotically uniform + extra order

Counting points in test sets, e.g. balls B_R

$$N_R := \sum_{i=1}^{N} \mathbb{1}_{B_R}(X_i)\ , \quad \text{where } (X_1, \dots, X_N) \sim \rho_V^{(N)}$$

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The **expected** number of points in B_R is

$$\mathbb{E}\left[N_R\right] \overset{th.}{\to} \rho |B_R|$$

Hyperuniformity in $\mathbb{R}^d$

The **variance** measures the rate of convergence.

Example: $(X_i)_i$ i.i.d. $\Rightarrow\ \mathbb{V}[N_R] \stackrel{th.}{\to} \rho|B_R|$.

Definition

$(\rho^{(N)})_{N\in\mathbb{N}}$ hyperuniform $\Longleftrightarrow \lim_{th.} \mathbb{V}[N_R] \sim |\partial B_R|$ for large R

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  $\Rightarrow\ R^{d-1}$-term cannot vanish.
- Hyperuniformity is a long-scale property.

Hyperuniformity on the sphere

Definition (Hyperuniformity)

Let $(X_N)_{N\in\mathbb{N}}$ be a sequence of point sets on the sphere $\mathbb{S}^d$. The *number variance* of the sequence for caps of opening angle ϕ is given by

$$V(X_N,\phi) = \mathbb{V}_{\mathbf{x}}\#\left(X_N \cap C(\mathbf{x},\phi)\right). \tag{1}$$

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- **hyperuniform for large caps** if

$$V(X_N, \phi) = o\left(N\right) \quad \text{as } N \to \infty \tag{2}$$

for all $\phi \in (0, \frac{\pi}{2})$;

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  1. $\lim_{N\to\infty}\phi_N = 0$
  2. $\lim_{N\to\infty} N\sigma(C(\cdot,\phi_N)) = \infty$.

- **hyperuniform for caps at threshold order**, if

$$\limsup_{N\to\infty} V(X_N, tN^{-\frac{1}{d}}) = \mathcal{O}(t^{d-1}) \quad \text{as } t\to\infty. \tag{4}$$

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- QMC-designs

Determinantal point process in $\mathbb{S}^2$

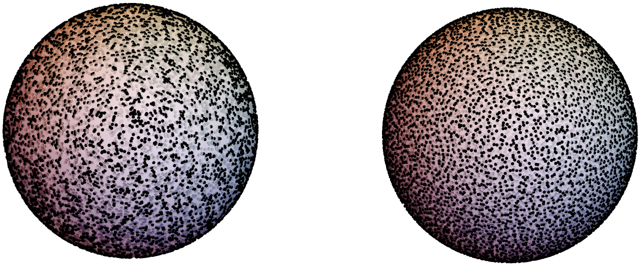

Figure: 10000 sampled points from an i.i.d. process and a DPP, resp.

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- Find explicit deterministic constructions for hyperuniform point sets for any N.
- Find explicit deterministic constructions for point sets achieving the best possible discrepancy bound (or even a bound better than $N^{-\frac{1}{2}}$)